Some subspaces of the k-th exterior power of a symplectic vector space

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Abstract

Let $\mathbb{K}$ be an arbitrary field, let n, k, l be nonnegative integers satisfying $n \geq 1$, $1 \leq k \leq 2n$, $0 \leq l \leq \min(n, k)$, and let V be a $2n$-dimensional vector space over $\mathbb{K}$ equipped with a nondegenerate alternating bilinear form f. Let $W_{k,l}$ denote the subspace of $\bigwedge^k V$ generated by all vectors $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k$, where $\bar{v}_1, \ldots, \bar{v}_k$ are k linearly independent vectors of V such that $\langle \bar{v}_1, \ldots, \bar{v}_l \rangle$ is totally isotropic with respect to f. We prove that $\dim(W_{k,l}) = \binom{2n}{k} - \binom{2n}{2l-k-2}$. We give a recursive method for constructing a basis of $W_{k,l}$ and give a decomposition of $W_{k,l}$ relative to a given hyperbolic basis of V. We also study two linear mappings, one between the spaces $W_{k,l}$ and $W_{k-2,l-1}$ and another one between $W_{k,l}$ and $W_{2n-k,n+l-k}$.

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1 Introduction

Let $\mathbb{K}$ be a field, let n be a strictly positive integer and let V be a $2n$-dimensional vector space over $\mathbb{K}$ equipped with a nondegenerate alternating bilinear form f. A subspace U of V is called *totally isotropic* (*with respect to* f) if $f(\bar{u}_1, \bar{u}_2) = 0$ for all $\bar{u}_1, \bar{u}_2 \in U$. If U is totally isotropic, then $\dim(U) \leq n$.

For every $k \in \{1, \ldots, 2n\}$, let $\bigwedge^k V$ be the k-th exterior power of V. Then $\dim(\bigwedge^k V) = \binom{2n}{k}$. The vector space $\bigwedge^k V$ is generated by all vectors of the form $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k$, where $\bar{v}_1, \ldots, \bar{v}_k$ are k (linearly independent) vectors of V. We can now ask the following question: for given $l \in \{0, \ldots, \min(n, k)\}$, what is the dimension of the subspace $W_{k,l}$ of $\bigwedge^k V$ generated by all vectors $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k$, where $\bar{v}_1, \ldots, \bar{v}_k$ are k (linearly independent) vectors of V such that $\langle \bar{v}_1, \ldots, \bar{v}_l \rangle$ is totally isotropic with respect to f. [Here, we use the following convention: if $l = 0$, then $\langle \bar{v}_1, \ldots, \bar{v}_l \rangle = \langle - \rangle := \{\bar{0}\}$. So, if $l \in \{0, 1\}$, then $W_{k,l} = \bigwedge^k V$.] The following theorem answers this question.

Theorem 1.1 *The dimension of $W_{k,l}$ is equal to $\binom{2n}{k} - \binom{2n}{2l-k-2}$.*

In the previous theorem, we used the convention that $\binom{a}{b} = 0$ for every $a \in \mathbb{N}$ and every $b \in \mathbb{Z} \setminus \{0, \ldots, a\}$. Note that Pascal's identity $\binom{a+1}{b+1} = \binom{a}{b+1} + \binom{a}{b}$ remains valid for every $a \in \mathbb{N}$ and every $b \in \mathbb{Z}$. In the special case that $1 \leq l = k \leq n$, Theorem 1.1 says that the dimension of the subspace of $\bigwedge^k V$ generated by all vectors of the form $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$ such that $\langle \bar{v}_1, \ldots, \bar{v}_k \rangle$ is totally isotropic is equal to $\binom{2n}{k} - \binom{2n}{k-2}$. This is probably known: a proof of this claim is contained in Bourbaki [1] for fields of characteristic 0, in Burau [2] for the field of the complex numbers (although the proof also extends to arbitrary fields of characteristic 0) and in Premet and Suprunenko [5] for arbitrary fields of odd characteristic. In De Bruyn [4], we also obtained that result for general fields in the special case that $k = l = n$ as an immediate consequence of the decomposition theorem for the natural embedding spaces for the symplectic dual polar spaces.

The proof of Theorem 1.1 which we will give allows us to construct in a recursive way a basis of the vector space $W_{k,l}$. Investigating the structure of the elements of such a basis, we are able to give a decomposition of the space $W_{k,l}$ relative to a given hyperbolic basis B of V. In the special case that $k = l = n$, this decomposition is precisely the decomposition of the natural embedding space of the symplectic dual polar space $DSp(2n, \mathbb{K})$ as stated in the main theorem of De Bruyn [4].

Among the tools which we will use to prove Theorem 1.1, there are two linear mappings, some of whose properties we will study. If $k \in \{3, \ldots, 2n\}$ and $1 \leq l \leq \min(k-1, n)$, then we will define and study a linear mapping between the spaces $W_{k,l}$ and $W_{k-2,l-1}$. If $k \in \{1, \ldots, 2n-1\}$ and $l \in \{\lceil \frac{k}{2} \rceil, \ldots, \min(n, k)\}$, then we will define a linear isomorphism between $W_{k,l}$ and $W_{2n-k,n+l-k}$.

2 A generating set of $W_{k,l}$

We continue with the notation introduced in Section 1. An ordered basis $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V is called a *hyperbolic basis* of V (*with respect to* f)

if $f(\bar{e}_i, \bar{e}_j) = f(\bar{f}_i, \bar{f}_j) = 0$ and $f(\bar{e}_i, \bar{f}_j) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$. If $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ is a hyperbolic basis of V, then

(1) for every permutation σ of $\{1, \ldots, n\}$, also $(\bar{e}_{\sigma(1)}, \bar{f}_{\sigma(1)}, \ldots, \bar{e}_{\sigma(n)}, \bar{f}_{\sigma(n)})$ is a hyperbolic basis of V;

(2) for every $\lambda \in \mathbb{K}^* := \mathbb{K} \setminus \{0\}$, also $(\frac{\bar{e}_1}{\lambda}, \lambda \bar{f}_1, \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n)$ is a hyperbolic basis of V;

(3) for every $\lambda \in \mathbb{K}$, also $(\bar{e}_1 + \lambda \bar{e}_2, \bar{f}_1, \bar{e}_2, -\lambda \bar{f}_1 + \bar{f}_2, \bar{e}_3, \bar{f}_3, \ldots, \bar{e}_n, \bar{f}_n)$ is a hyperbolic basis of V;

(4) for every $\lambda \in \mathbb{K}$, also $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_{n-1}, \bar{f}_{n-1}, \bar{e}_n, \bar{f}_n + \lambda \bar{e}_n)$ is a hyperbolic basis of V;

(5) for every $\lambda \in \mathbb{K}$, also $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_{n-1}, \bar{f}_{n-1}, \bar{e}_n + \lambda \bar{f}_n, \bar{f}_n)$ is a hyperbolic basis of V.

For every $i \in \{1, 2, 3, 4, 5\}$, let Ω_i denote the set of all ordered pairs (B_1, B_2) of hyperbolic bases of V such that B_2 can be obtained from B_1 as described in (i) above.

Lemma 2.1 *If B and B' are two hyperbolic bases of V, then there exist hyperbolic bases $B_0, B_1, \ldots, B_k$ $(k \geq 0)$ of V such that $B_0 = B$, $B_k = B'$ and $(B_{i-1}, B_i) \in \Omega_1 \cup \cdots \cup \Omega_5$ for every $i \in \{1, \ldots, k\}$.*

Proof. Put $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ and $B' = (\bar{e}'_1, \bar{f}'_1, \ldots, \bar{e}'_n, \bar{f}'_n)$. Put $E = \langle \bar{e}_1, \ldots, \bar{e}_n \rangle$, $E' = \langle \bar{e}'_1, \ldots, \bar{e}'_n \rangle$, $F = \langle \bar{f}_1, \ldots, \bar{f}_n \rangle$ and $F' = \langle \bar{f}'_1, \ldots, \bar{f}'_n \rangle$. The proof of the lemma will occur in 3 steps.

(1) Suppose $E = E'$ and $F = F'$. Since the maps $(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n) \mapsto (\bar{g}_{\sigma(1)}, \bar{g}_{\sigma(2)}, \ldots, \bar{g}_{\sigma(n)})$, $(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n) \mapsto (\frac{\bar{g}_1}{\lambda}, \bar{g}_2, \ldots, \bar{g}_n)$ and $(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n) \mapsto (\bar{g}_1 + \lambda \bar{g}_2, \bar{g}_2, \ldots, \bar{g}_n)$ allow us to transform any basis of E to any other basis of E, there exist hyperbolic bases $B_0, B_1, \ldots, B_k$ $(k \geq 0)$ of V such that (i) $B_0 = B$, (ii) $(B_{i-1}, B_i) \in \Omega_1 \cup \Omega_2 \cup \Omega_3$ for every $i \in \{1, \ldots, k\}$, and (iii) B_k is of the form $(\bar{e}'_1, \bar{f}''_1, \ldots, \bar{e}'_n, \bar{f}''_n)$ with $F = \langle \bar{f}''_1, \ldots, \bar{f}''_n \rangle$. The vector $\bar{f}''_i$, $i \in \{1, \ldots, n\}$, is uniquely determined by the vectors $\bar{e}'_1, \ldots, \bar{e}'_n$: it is the unique vector of F which is f-orthogonal with every $\bar{e}'_j$, $j \neq i$, and which satisfies $(\bar{e}'_i, \bar{f}''_i) = 1$. Hence, $\bar{f}''_i = \bar{f}'_i$ for every $i \in \{1, \ldots, k\}$, i.e. $B_k = B'$.

(2) Suppose ($E = E'$ and $\dim(F \cap F') = n - 1$) or ($F = F'$ and $\dim(E \cap E') = n - 1$). We will only treat the case $E = E'$ and $\dim(F \cap F') = n - 1$, since the other case is completely similar. By (1), the lemma will hold for the pair (B, B') as soon as it holds for at least one pair (B, B') giving rise to the same subspaces $E = E'$, F and F'. So, without loss of generality, we may suppose that B and B' are in such a way that $\{\bar{f}_1, \ldots, \bar{f}_{n-1}\}$ is a basis of $F \cap F'$ and $\bar{e}'_i = \bar{e}_i$ for every $i \in \{1, \ldots, n\}$. Then $\bar{f}'_i = \bar{f}_i$ for every

$i \in \{1, \ldots, n-1\}$ and there exists a $\lambda \in \mathbb{K}^*$ such that $\bar{f}'_n = \bar{f}_n + \lambda \bar{e}_n$. So, $(B, B') \in \Omega_4$.

(3) Consider the following graph Γ. The vertices of Γ are the pairs (X, Y) where X and Y are two complementary totally isotropic n-dimensional subspaces of V. Two vertices (X, Y) and (X', Y') of Γ are adjacent if either ($X = X'$ and $\dim(Y \cap Y') = n-1$) or ($Y = Y'$ and $\dim(X \cap X') = n-1$). We will now prove that the graph Γ is connected. This fact, combined with (1) and (2), then finishes the proof of the lemma. We will use some terminology from Incidence Geometry to establish the connectedness of Γ. With the pair (V, f), there is associated the dual polar space $DSp(2n, \mathbb{K})$, that is the point-line geometry with points the n-dimensional totally isotropic subspaces of V and with lines all the sets of n-dimensional totally isotropic subspaces which contain a given $(n-1)$-dimensional totally isotropic subspace. The distance $\mathrm{d}(U_1, U_2)$ between two points U_1 and U_2 of $DSp(2n, \mathbb{K})$ in the collinearity graph Δ of $DSp(2n, \mathbb{K})$ is equal to $n - \dim(U_1 \cap U_2)$. The dual polar space $DSp(2n, \mathbb{K})$ is a so-called *near polygon* (see [3]) which means that for every point-line pair (x, L), there exists a unique point $\pi_L(x)$ on L nearest to x (w.r.t. the distance in Δ). If U is a totally isotropic subspace, then the set of all n-dimensional totally isotropic subspaces containing U is a convex subspace of diameter $n - \dim(U)$. [This is a set P of points satisfying the following properties: (i) if x and y are two distinct collinear points of $DSp(2n, \mathbb{K})$, then every point of the unique line through x and y also belongs to P; (ii) if x and y are two points belonging to P, then also every point on a shortest path (in Δ) between x and y belongs to P; (iii) the maximal distance (in Δ) between two points of P is equal to $n - \dim(U)$.] Notice that the vertices of Γ are the pairs (x, y) of opposite points of $DSp(2n, \mathbb{K})$. We will now prove by induction on $\mathrm{d}(x_1, x_2)$ that any two vertices (x_1, y_1) and (x_2, y_2) of Γ are connected by a path.

Suppose first that $\mathrm{d}(x_1, x_2) = 0$, i.e. $x_1 = x_2$. Then the claim follows from the fact that the subgraph of Δ induced on the set of points opposite to a given vertex is connected, see e.g. [3, Theorem 2.7].

Suppose $\mathrm{d}(x_1, x_2) \geq 1$. Let x_3 be a point collinear with x_2 at distance $\mathrm{d}(x_1, x_2) - 1$ from x_1. By the induction hypothesis, it suffices to show that there exists a point y_3 opposite to x_3 such that (x_2, y_2) and (x_3, y_3) are contained in the same connected component of Γ. This clearly holds if $\mathrm{d}(x_3, y_2) = n$. (Take $y_3 = y_2$.) Suppose therefore that $\mathrm{d}(x_3, y_2) = n-1$. Let L denote a line through y_2 which is not contained in the unique convex subspace of diameter $n-1$ containing x_3 and y_2, and let y_3 be a point of $L \setminus \{y_2\}$ distinct from $\pi_L(x_2)$. Then $\mathrm{d}(x_2, y_3) = \mathrm{d}(x_3, y_3) = n$. So, $(x_2, y_2) \sim_\Gamma (x_2, y_3) \sim_\Gamma (x_3, y_3)$. This is precisely what we needed to show. $\square$

Now, let $k \in \{1, \ldots, 2n\}$ and $l \in \{0, \ldots, \min(k, n)\}$. Recall that $W_{k,l}$ is the subspace of $\bigwedge^k V$ generated by all vectors of the form $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$ where $\langle \bar{v}_1, \ldots, \bar{v}_l \rangle$ is totally isotropic with respect to f. Obviously, if $l' = \min(n, k)$, then $W_{k,l'} \subseteq W_{k,l'-1} \subseteq \cdots \subseteq W_{k,1} = W_{k,0} = \bigwedge^k V$. We will now determine $W_{2,2}$ for every $n \geq 2$.

Lemma 2.2 *If $n \geq 2$, then the dimension of $W_{2,2}$ is equal to $2n^2 - n - 1$.*

Proof. Let $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ be an arbitrary hyperbolic basis of V. Then obviously, the following linearly independent vectors belong to $W_{2,2}$:
(i) $\bar{e}_i \wedge \bar{e}_j$, $1 \leq i < j \leq n$;
(ii) $\bar{f}_i \wedge \bar{f}_j$, $1 \leq i < j \leq n$;
(iii) $\bar{e}_i \wedge \bar{f}_j$, $i, j \in \{1, \ldots, n\}$ with $i \neq j$;
(iv) $\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_i \wedge \bar{f}_i = (\bar{e}_1 + \bar{e}_i) \wedge (\bar{f}_1 - \bar{f}_i) + \bar{e}_1 \wedge \bar{f}_i - \bar{e}_i \wedge \bar{f}_1$, $i \in \{2, \ldots, n\}$.

These are $\frac{n(n-1)}{2} + \frac{n(n-1)}{2} + (n^2 - n) + (n - 1) = 2n^2 - n - 1$ vectors. Now, an arbitrary vector α of $W_{2,2}$ is of the form

$$(a_1\bar{e}_1 + b_1\bar{f}_1 + \cdots + a_n\bar{e}_n + b_n\bar{f}_n) \wedge (a'_1\bar{e}_1 + b'_1\bar{f}_1 + \cdots + a'_n\bar{e}_n + b'_n\bar{f}_n),$$

where $f(a_1\bar{e}_1 + \cdots + b_n\bar{f}_n, a'_1\bar{e}_1 + \cdots + b'_n\bar{f}_n) = \sum_{i=1}^n (a_i b'_i - b_i a'_i) = 0$. It is straightforward to verify that such a vector α can be written as a linear combination of the vectors mentioned in (i)–(iv) above. □

Lemma 2.3 *Let $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$ be k linearly independent vectors of V such that $\langle \bar{v}_1, \ldots, \bar{v}_l \rangle$ is totally isotropic with respect to f. Then there exists a hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V, a $\lambda \in \mathbb{K}^*$, and an $m \in \{\max(0, k - n), \ldots, \min(k - l, \lfloor \frac{k}{2} \rfloor)\}$ such that*

$$\bar{v}_1 \wedge \cdots \wedge \bar{v}_k = \lambda \cdot (\bar{e}_1 \wedge \bar{f}_1) \wedge \cdots \wedge (\bar{e}_m \wedge \bar{f}_m) \wedge \bar{e}_{m+1} \wedge \cdots \wedge \bar{e}_{k-m}.$$

Proof. Let R denote the radical of the alternating bilinear form of $\langle \bar{v}_1, \ldots, \bar{v}_k \rangle$ induced by f and let U be a subspace of $\langle \bar{v}_1, \ldots, \bar{v}_k \rangle$ complementary to R. The alternating bilinear form f_U of U induced by f is nondegenerate. Hence, $\dim(U)$ is even, say $\dim(U) = 2m \geq 0$. Let $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_m, \bar{f}_m)$ be a hyperbolic basis of U with respect to f_U. Consider a basis $\{\bar{e}_{m+1}, \ldots, \bar{e}_{k-m}\}$ of R. Then $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_m, \bar{f}_m, \bar{e}_{m+1}, \ldots, \bar{e}_{k-m})$ can be extended to a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V. Obviously, $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k = \lambda \cdot (\bar{e}_1 \wedge \bar{f}_1) \wedge \cdots \wedge (\bar{e}_m \wedge \bar{f}_m) \wedge \bar{e}_{m+1} \wedge \cdots \wedge \bar{e}_{k-m}$ for some $\lambda \in \mathbb{K}^*$. Clearly, $\dim(U) = 2m \leq k$, i.e. $m \leq \lfloor \frac{k}{2} \rfloor$. Also, since the dimension of a maximal totally isotropic subspace of $\langle \bar{v}_1, \ldots, \bar{v}_k \rangle = \langle R, U \rangle$ is equal to $k - m$, we have $l \leq k - m \leq n$, i.e. $k - n \leq m \leq k - l$. □

For every hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \dots, \bar{e}_n, \bar{f}_n)$ of V, let $\mathcal{G}(B, k, l)$ denote the set of all vectors

$$(\bar{e}_{\sigma(1)} \wedge \bar{f}_{\sigma(1)} - \bar{e}_{\sigma(2)} \wedge \bar{f}_{\sigma(2)}) \wedge \cdots \wedge (\bar{e}_{\sigma(2m-1)} \wedge \bar{f}_{\sigma(2m-1)} - \bar{e}_{\sigma(2m)} \wedge \bar{f}_{\sigma(2m)}) \wedge$$

$$\bar{g}_{\sigma(2m+1)} \wedge \cdots \wedge \bar{g}_{\sigma(2l'-k)} \wedge (\bar{e}_{\sigma(2l'-k+1)} \wedge \bar{f}_{\sigma(2l'-k+1)}) \wedge \cdots \wedge (\bar{e}_{\sigma(l')} \wedge \bar{f}_{\sigma(l')}),$$

where (i) $l' \in \mathbb{N}$ such that $\max(\lceil \frac{k}{2} \rceil, l) \leq l' \leq \min(n, k)$, (ii) $m \in \mathbb{N}$ such that $0 \leq 2m \leq 2l' - k$, (iii) σ is a permutation of $\{1, \dots, n\}$, and (iv) $\bar{g}_{\sigma(i)} \in \{\bar{e}_{\sigma(i)}, \bar{f}_{\sigma(i)}\}$ for every $i \in \{2m+1, \dots, 2l'-k\}$. Let $W_{k,l}(B)$ denote the subspace of $\bigwedge^k V$ generated by all vectors of $\mathcal{G}(B, k, l)$. One readily verifies that if B_1 and B_2 are two hyperbolic bases of V such that $(B_1, B_2) \in \Omega_1 \cup \cdots \cup \Omega_5$, then $W_{k,l}(B_1) = W_{k,l}(B_2)$. Hence by Lemma 2.1, $W_{k,l}(B)$ only depends on k and l and not on the hyperbolic basis B of V.

Lemma 2.4 *For every hyperbolic basis B of V, $W_{k,l}(B) = W_{k,l}$.*

Proof. (1) We prove that $W_{k,l}(B) \subseteq W_{k,l}$.

Put $B = (\bar{e}_1, \bar{f}_1, \dots, \bar{e}_n, \bar{f}_n)$ and consider one of the above vectors:

$$(\bar{e}_{\sigma(1)} \wedge \bar{f}_{\sigma(1)} - \bar{e}_{\sigma(2)} \wedge \bar{f}_{\sigma(2)}) \wedge \cdots \wedge (\bar{e}_{\sigma(2m-1)} \wedge \bar{f}_{\sigma(2m-1)} - \bar{e}_{\sigma(2m)} \wedge \bar{f}_{\sigma(2m)}) \wedge$$

$$\bar{g}_{\sigma(2m+1)} \wedge \cdots \wedge \bar{g}_{\sigma(2l'-k)} \wedge (\bar{e}_{\sigma(2l'-k+1)} \wedge \bar{f}_{\sigma(2l'-k+1)}) \wedge \cdots \wedge (\bar{e}_{\sigma(l')} \wedge \bar{f}_{\sigma(l')}).$$

Since $\bar{e}_{\sigma(2i-1)} \wedge \bar{f}_{\sigma(2i-1)} - \bar{e}_{\sigma(2i)} \wedge \bar{f}_{\sigma(2i)} = (\bar{e}_{\sigma(2i-1)} + \bar{e}_{\sigma(2i)}) \wedge (\bar{f}_{\sigma(2i-1)} - \bar{f}_{\sigma(2i)}) + \bar{e}_{\sigma(2i-1)} \wedge \bar{f}_{\sigma(2i)} - \bar{e}_{\sigma(2i)} \wedge \bar{f}_{\sigma(2i-1)}$ for every $i \in \{1, \dots, m\}$, this vector can be written as a sum of 3^m other vectors. It suffices to show that each of these 3^m other vectors belongs to $W_{k,l}$. So, let

$$(\bar{v}_1 \wedge \bar{v}_2) \wedge (\bar{v}_3 \wedge \bar{v}_4) \wedge \cdots \wedge (\bar{v}_{2m-1} \wedge \bar{v}_{2m}) \wedge$$

$$\bar{g}_{\sigma(2m+1)} \wedge \cdots \wedge \bar{g}_{\sigma(2l'-k)} \wedge (\bar{e}_{\sigma(2l'-k+1)} \wedge \bar{f}_{\sigma(2l'-k+1)}) \wedge \cdots \wedge (\bar{e}_{\sigma(l')} \wedge \bar{f}_{\sigma(l')})$$

be one of these 3^m vectors. Here, $\bar{v}_{2i-1}, \bar{v}_{2i} \in \langle \bar{e}_{\sigma(2i-1)}, \bar{f}_{\sigma(2i-1)}, \bar{e}_{\sigma(2i)}, \bar{f}_{\sigma(2i)} \rangle$ and $f(\bar{v}_{2i-1}, \bar{v}_{2i}) = 0$ for every $i \in \{1, \dots, m\}$. Obviously, the above vector belongs to $W_{k,l}$ since $l' \geq l$ and $\langle \bar{v}_1, \dots, \bar{v}_{2m}, \bar{g}_{\sigma(2m+1)}, \dots, \bar{g}_{\sigma(2l'-k)}, \bar{e}_{\sigma(2l'-k+1)}, \dots, \bar{e}_{\sigma(l')} \rangle$ is totally isotropic with respect to f.

(2) We prove that $W_{k,l} \subseteq W_{k,l}(B)$.

We must prove that $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k \in W_{k,l}(B)$ where $\bar{v}_1, \dots, \bar{v}_k$ are k linearly independent vectors of V such that $\langle \bar{v}_1, \dots, \bar{v}_l \rangle$ is totally isotropic with respect to f. By Lemma 2.3, there exists a hyperbolic basis $B' = (\bar{e}'_1, \bar{f}'_1, \dots, \bar{e}'_n, \bar{f}'_n)$, a $\lambda \in \mathbb{K}^*$ and an $m \in \{\max(0, k-n), \dots, \min(k-l, \lfloor \frac{k}{2} \rfloor)\}$ such that $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k = \lambda \cdot (\bar{e}'_1 \wedge \bar{f}'_1) \wedge \cdots \wedge (\bar{e}'_m \wedge \bar{f}'_m) \wedge \bar{e}'_{m+1} \wedge \cdots \wedge \bar{e}'_{k-m}$. Now, $(\bar{e}'_1 \wedge \bar{f}'_1) \wedge \cdots \wedge (\bar{e}'_m \wedge \bar{f}'_m) \wedge \bar{e}'_{m+1} \wedge \cdots \wedge \bar{e}'_{k-m}$ is a vector of $\mathcal{G}(B', k, l)$. In order to prove this claim, we must verify that $\max(\lceil \frac{k}{2} \rceil, l) \le k - m \le \min(n, k)$. But this is clear: since $k - n \le m \le k - l$, we have $l \le k - m \le n$, and since $m \le \frac{k}{2}$, we have $\frac{k}{2} \le k - m$.

Since $(\bar{e}'_1 \wedge \bar{f}'_1) \wedge \cdots \wedge (\bar{e}'_m \wedge \bar{f}'_m) \wedge \bar{e}'_{m+1} \wedge \cdots \wedge \bar{e}'_{k-m}$ is a vector of $\mathcal{G}(B', k, l)$, we have $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k \in W_{k,l}(B') = W_{k,l}(B)$. $\square$

Lemma 2.5 *If $0 \le l \le \lceil \frac{k}{2} \rceil$, then $W_{k,l} = \bigwedge^k V$.*

Proof. By Lemma 2.3, if $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$ are k linearly independent vectors of V, then there exists a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$, a $\lambda \in \mathbb{K}^*$ and an $m \in \{\max(0, k-n), \ldots, \lfloor \frac{k}{2} \rfloor\}$ such that $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k = \lambda \cdot (\bar{e}_1 \wedge \bar{f}_1) \wedge \cdots \wedge (\bar{e}_m \wedge \bar{f}_m) \wedge \bar{e}_{m+1} \wedge \cdots \wedge \bar{e}_{k-m}$. Hence, $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k \in W_{k,k-m} \subseteq W_{k,l}$ since $k - m \ge k - \lfloor \frac{k}{2} \rfloor = \lceil \frac{k}{2} \rceil \ge l$. Since this holds for any k linearly independent vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$ of V, $W_{k,l} = \bigwedge^k V$. $\square$

By Lemmas 2.2 and 2.5, we have

Corollary 2.6 *If $k \le 2$ or $l \le \lceil \frac{k}{2} \rceil$, then* $\dim(W_{k,l}) = \binom{2n}{k} - \binom{2n}{2l-k-2}$.

3 Bounds for the dimension of $W_{k,l}$

We continue with the notation introduced in Sections 1 and 2.

Definition. Let $A(n, k, l, \mathbb{K})$ denote the dimension of the vector space $W_{k,l}$.

Now, let $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ be a hyperbolic basis of V and let $\mathcal{B}_k(B)$ denote the basis of $\bigwedge^k V$ whose elements are the vectors $\bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_j} \wedge \bar{f}_{i'_1} \wedge \cdots \wedge \bar{f}_{i'_{k-j}}$ where $i_1, \ldots, i_j, i'_1, \ldots, i'_{k-j} \in \{1, \ldots, n\}$ such that $i_1 < \cdots < i_j$ and $i'_1 < \cdots < i'_{k-j}$. We call $\mathcal{B}_k(B)$ the *standard basis of $\bigwedge^k V$ with respect to B*. Put $\mathcal{G}(B, k, l) = \{\alpha_1, \ldots, \alpha_N\}$, and let M denote the $N \times \binom{2n}{k}$-matrix whose i-th row ($i \in \{1, \ldots, N\}$) is the coordinate vector of α_i with respect to some given ordering of the basis $\mathcal{B}_k(B)$. Then any entry of M is equal to either 0, 1 or -1. The dimension $A(n, k, l, \mathbb{K})$ of $W_{k,l}$ is equal to the largest size of a nonsingular square submatrix of M. The following is now obvious:

Lemma 3.1 • *If $\mathbb{K}_1$ and $\mathbb{K}_2$ are two fields of the same characteristic, then $A(n, k, l, \mathbb{K}_1) = A(n, k, l, \mathbb{K}_2)$ for any integers n, k and l satisfying $n \ge 1$, $1 \le k \le 2n$ and $l \in \{0, \ldots, \min(k, n)\}$.*

• *If $\mathbb{K}_1$ and $\mathbb{K}_2$ are two fields such that $\mathbb{K}_2$ has characteristic 0, then $A(n, k, l, \mathbb{K}_1) \le A(n, k, l, \mathbb{K}_2)$ for any integers n, k and l satisfying $n \ge 1$, $1 \le k \le 2n$ and $l \in \{0, \ldots, \min(k, n)\}$.*

Lemma 3.2 $A(n,k,l,\mathbb{K}) \geq \binom{2n}{k} - \binom{2n}{2l-k-2}$.

Proof. We will prove this by induction on k. If $k \leq 2$ or $l \leq \lceil \frac{k}{2} \rceil$, then the lemma holds by Corollary 2.6. So, in the sequel, we will suppose that $k \geq 3$ and $l > \lceil \frac{k}{2} \rceil$. Since $l \leq n$, this implies that $n \geq l \geq 3$ and $k \not\in \{2n-1, 2n\}$.

Let $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ be a given hyperbolic basis of V, let V' be the subspace $\langle \bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n \rangle$ of V and let f' be the nondegenerate alternating bilinear form of V' induced by f. Then $B' = (\bar{e}_2, \bar{f}_2, \ldots, \bar{e}_n, \bar{f}_n)$ is a hyperbolic basis of V' with respect to f'. For nonnegative integers k', l' satisfying $1 \leq k' \leq 2n-2$ and $0 \leq l' \leq \min(k', n-1)$, let $W'_{k',l'}$ denote the subspace of $\bigwedge^{k'} V'$ generated by all vectors $\bar{v}_1 \wedge \cdots \wedge \bar{v}_{k'}$ where $\bar{v}_1, \ldots, \bar{v}_{k'}$ are k' vectors of V' such that $\langle \bar{v}_1, \ldots, \bar{v}_{l'} \rangle$ is totally isotropic with respect to f'.

Every vector χ of $\bigwedge^k V$ can be written in a unique way as $\bar{e}_1 \wedge \bar{f}_1 \wedge \alpha(\chi) + \bar{e}_1 \wedge \beta(\chi) + \bar{f}_1 \wedge \gamma(\chi) + \delta(\chi)$, where $\alpha(\chi) \in \bigwedge^{k-2} V'$, $\beta(\chi) \in \bigwedge^{k-1} V'$, $\gamma(\chi) \in \bigwedge^{k-1} V'$ and $\delta(\chi) \in \bigwedge^k V'$. Let θ denote the linear map from $W_{k,l}$ to $\bigwedge^{k-2} V'$ mapping each $\chi \in W_{k,l}$ to $\alpha(\chi)$. We have

$$\dim(W_{k,l}) = \dim(Im(\theta)) + \dim(ker(\theta)). \quad (1)$$

We will now determine lower bounds for $\dim(Im(\theta))$ and $\dim(ker(\theta))$.

(i) If β is one of the vectors of $\mathcal{G}(B', k-1, l-1)$, then $\bar{e}_1 \wedge \beta$ and $\bar{f}_1 \wedge \beta$ are vectors of $\mathcal{G}(B,k,l)$. Hence, $\bar{e}_1 \wedge W'_{k-1,l-1} \subseteq W_{k,l}$ and $\bar{f}_1 \wedge W'_{k-1,l-1} \subseteq W_{k,l}$. Moreover, these subspaces belong to $ker(\theta)$ and have dimension at least $\binom{2n-2}{k-1} - \binom{2n-2}{2l-k-3}$ by the induction hypothesis.

(ii) Suppose $l \leq n-1$. If δ is one of the vectors of $\mathcal{G}(B',k,l)$, then δ is also one of the vectors of $\mathcal{G}(B,k,l)$. Hence, $W'_{k,l} \subseteq W_{k,l}$. The subspace $W'_{k,l}$ belongs to $ker(\theta)$ and has dimension at least $\binom{2n-2}{k} - \binom{2n-2}{2l-k-2}$ by the induction hypothesis.

(iii) If $l = n$, then $\binom{2n-2}{k} - \binom{2n-2}{2l-k-2} = 0$.

By (i), (ii) and (iii),

$$\dim(ker(\theta)) \geq 2 \cdot \binom{2n-2}{k-1} - 2 \cdot \binom{2n-2}{2l-k-3} + \binom{2n-2}{k} - \binom{2n-2}{2l-k-2}. \quad (2)$$

(iv) We prove that $Im(\theta) = W'_{k-2,l-2}$. One readily observes that any vector of $\mathcal{G}(B,k,l)$ is mapped by θ to either the zero vector or a vector of $\mathcal{G}(B', k-2, l-2)$. Hence, $Im(\theta) \subseteq W'_{k-2,l-2}$. We will now also prove that $W'_{k-2,l-2} \subseteq Im(\theta)$. Let α be one of the vectors of $\mathcal{G}(B', k-2, l-2)$.

If the number of factors of the form $\bar{e}_i \wedge \bar{f}_i$ (let's call them bad factors) in the expression of α is less than $k-l$, then $\chi := \bar{e}_1 \wedge \bar{f}_1 \wedge \alpha$ is a vector of $\mathcal{G}(B,k,l)$.

Suppose now that the number of bad factors in the expression of α is equal to $k-l$. Then the number of $i \in \{2,\dots,n\}$ such that $\bar{e}_i$ or $\bar{f}_i$ occurs in the expression of α is equal to $l-2 \leq n-2$. Hence, there exists an $i^* \in \{2,\dots,n\}$ such that neither $\bar{e}_{i^*}$ nor $\bar{f}_{i^*}$ occurs in the expression of α. Then $\chi := (\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_{i^*} \wedge \bar{f}_{i^*}) \wedge \alpha$ is a vector of $\mathcal{G}(B,k,l)$.

In either case, we have $\theta(\chi) = \alpha$. Hence, $W'_{k-2,l-2} \subseteq Im(\theta)$ since α was an arbitrary element of the generating set $\mathcal{G}(B',k-2,l-2)$ of $W'_{k-2,l-2}$.

By (iv) and the induction hypothesis, we have

$$\dim(Im(\theta)) \geq \binom{2n-2}{k-2} - \binom{2n-2}{2l-k-4}. \tag{3}$$

By (1), (2) and (3) and Pascal's identity, we have $A(n,k,l,\mathbb{K}) = \dim(W_{k,l}) \geq \binom{2n}{k} - \binom{2n}{2l-k-2}$. $\square$

4 Two linear mappings

(I) For every hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \dots, \bar{e}_n, \bar{f}_n)$ of V and every $k \in \{2,\dots,2n\}$, we now define a linear map $\theta_{k,B} : \bigwedge^k V \to \bigwedge^{k-2} V$. If $m \in \mathbb{N}$ such that $\max(k-n,0) \leq m \leq \lfloor \frac{k}{2} \rfloor$, if σ is a permutation of $\{1,\dots,n\}$ and if $\bar{g}_{\sigma(i)} \in \{\bar{e}_{\sigma(i)}, \bar{f}_{\sigma(i)}\}$ for every $i \in \{m+1,\dots,k-m\}$, then we define

$$\theta_{k,B}\Big[(\bar{e}_{\sigma(1)} \wedge \bar{f}_{\sigma(1)}) \wedge \cdots \wedge (\bar{e}_{\sigma(m)} \wedge \bar{f}_{\sigma(m)}) \wedge \bar{g}_{\sigma(m+1)} \wedge \cdots \wedge \bar{g}_{\sigma(k-m)}\Big] \;:=$$

$$\sum_{i=1}^{m} (\bar{e}_{\sigma(1)} \wedge \bar{f}_{\sigma(1)}) \wedge \cdots \wedge (\widehat{\bar{e}_{\sigma(i)} \wedge \bar{f}_{\sigma(i)}}) \wedge \cdots \wedge (\bar{e}_{\sigma(m)} \wedge \bar{f}_{\sigma(m)}) \wedge \bar{g}_{\sigma(m+1)} \wedge \cdots \wedge \bar{g}_{\sigma(k-m)}.$$

[If $m=0$, then $\theta_{k,B}[\cdots] = 0$; if $m=1$ and $k=2$, then $\theta_{k,B}[\cdots] = 1$.] Since this defines $\theta_{k,B}$ for every element of the standard basis of $\bigwedge^k V$ with respect to B, we have defined $\theta_{k,B}$ for all vectors of $\bigwedge^k V$. It is straightforward (but perhaps tedious) to verify that if B_1 and B_2 are two hyperbolic bases of V such that $(B_1,B_2) \in \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_5$, then $\theta_{k,B_1} = \theta_{k,B_2}$. Hence by Lemma 2.1, there exists a linear map $\theta_k : \bigwedge^k V \to \bigwedge^{k-2} V$ such that $\theta_{k,B} = \theta_k$ for every hyperbolic basis B of V.

(II) For every hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \dots, \bar{e}_n, \bar{f}_n)$ of V and every $k \in \{1,\dots,2n-1\}$, we now define a linear isomorphism $\phi_{k,B} : \bigwedge^k V \to \bigwedge^{2n-k} V$.

If $m \in \mathbb{N}$ such that $\max(k-n,0) \leq m \leq \lfloor \frac{k}{2} \rfloor$, if σ is a permutation of $\{1,\ldots,n\}$ and if $\bar{g}_{\sigma(i)} \in \{\bar{e}_{\sigma(i)}, \bar{f}_{\sigma(i)}\}$ for every $i \in \{m+1,\ldots,k-m\}$, then we define

$$\phi_{k,B}\Big[(\bar{e}_{\sigma(1)} \wedge \bar{f}_{\sigma(1)}) \wedge \cdots \wedge (\bar{e}_{\sigma(m)} \wedge \bar{f}_{\sigma(m)}) \wedge \bar{g}_{\sigma(m+1)} \wedge \cdots \wedge \bar{g}_{\sigma(k-m)}\Big] :=$$

$$(-1)^m \cdot \bar{g}_{\sigma(m+1)} \wedge \cdots \wedge \bar{g}_{\sigma(k-m)} \wedge (\bar{e}_{\sigma(k-m+1)} \wedge \bar{f}_{\sigma(k-m+1)}) \wedge \cdots \wedge (\bar{e}_{\sigma(n)} \wedge \bar{f}_{\sigma(n)}).$$

Since this defines $\phi_{k,B}$ for every element of the standard basis of $\bigwedge^k V$ with respect to B, we have defined $\phi_{k,B}$ for all vectors of $\bigwedge^k V$. It is straightforward to verify that if B_1 and B_2 are two hyperbolic bases of V such that $(B_1, B_2) \in \Omega_1 \cup \cdots \cup \Omega_5$, then $\phi_{k,B_1} = \phi_{k,B_2}$. Hence by Lemma 2.1, there exists a linear isomorphism $\phi_k : \bigwedge^k V \to \bigwedge^{2n-k} V$ such that $\phi_{k,B} = \phi_k$ for every hyperbolic basis B of V. Obviously, $\phi_{2n-k} \circ \phi_k(\alpha) = (-1)^{n-k} \cdot \alpha$ for every $\alpha \in \bigwedge^k V$.

Lemma 4.1 *If $k \in \{1,\ldots,2n-1\}$ and $l \in \{\lceil \frac{k}{2} \rceil, \ldots, \min(n,k)\}$, then $\phi_k(W_{k,l}) = W_{2n-k,n+l-k}$.*

Proof. It suffices to prove that $\phi_k(W_{k,l}) \subseteq W_{2n-k,n+l-k}$, or equivalently, that $\phi_k(\bar{v}_1 \wedge \cdots \wedge \bar{v}_k) \in W_{2n-k,n+l-k}$ for all k linearly independent vectors $\bar{v}_1, \ldots, \bar{v}_k$ of V such that $\langle \bar{v}_1, \ldots, \bar{v}_l \rangle$ is totally isotropic with respect to f. [Notice that if $l \in \{\lceil \frac{k}{2} \rceil, \ldots, \min(n,k)\}$, then also $n+l-k \in \{\lceil \frac{2n-k}{2} \rceil, \ldots, \min(n, 2n-k)\}$. By symmetry, we would then also have $\phi_{2n-k}(W_{2n-k,n+l-k}) \subseteq W_{k,l}$. It then follows that $\phi_k(W_{k,l}) = W_{2n-k,n+l-k}$ and $\phi_{2n-k}(W_{2n-k,n+l-k}) = W_{k,l}$.] By Lemma 2.3, there exists a hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V, a $\lambda \in \mathbb{K}^*$ and an $m \in \{\max(0,k-n), \ldots, \min(k-l, \lfloor \frac{k}{2} \rfloor)\}$ such that $\bar{v}_1 \wedge \cdots \wedge \bar{v}_k = \lambda \cdot (\bar{e}_1 \wedge \bar{f}_1) \wedge \cdots \wedge (\bar{e}_m \wedge \bar{f}_m) \wedge \bar{e}_{m+1} \wedge \cdots \wedge \bar{e}_{k-m}$. Then $\phi_k(\bar{v}_1 \wedge \cdots \wedge \bar{v}_k) = \phi_{k,B}(\bar{v}_1 \wedge \cdots \wedge \bar{v}_k) = (-1)^m \lambda \cdot \bar{e}_{m+1} \wedge \cdots \wedge \bar{e}_{k-m} \wedge (\bar{e}_{k-m+1} \wedge \bar{f}_{k-m+1}) \wedge \cdots \wedge (\bar{e}_n \wedge \bar{f}_n)$. This vector belongs to $W_{2n-k,n-m}$ and hence also to $W_{2n-k,n+l-k}$ since $n+l-k \leq n-m$. $\square$

Lemma 4.2 *If $k \in \{2,\ldots,n\}$, then $W_{k,k} \subseteq ker(\theta_k)$.*

Proof. Let $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$ be k linearly independent vectors of V such that $\langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \rangle$ is totally isotropic with respect to f. Then there exists a hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V such that $\bar{v}_i = \bar{e}_i$ for every $i \in \{1,\ldots,k\}$. Then $\theta_k(\bar{v}_1 \wedge \cdots \wedge \bar{v}_k) = \theta_{k,B}(\bar{e}_1 \wedge \cdots \wedge \bar{e}_k) = 0$. $\square$

Lemma 4.3 *If $k \geq 3$ and $l \in \{1,\ldots,\min(k-1,n)\}$, then $\theta_k(W_{k,l}) \subseteq W_{k-2,l-1}$.*

Proof. Let $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ be an arbitrary hyperbolic basis of V. If $\alpha \in \mathcal{G}(B,k,l)$, then one readily verifies that $\theta_k(\alpha) = \theta_{k,B}(\alpha)$ is a sum of vectors belonging to $\mathcal{G}(B,k-2,l-1)$. This implies that $\theta_k(W_{k,l}) \subseteq W_{k-2,l-1}$. $\square$

Lemma 4.4 *Let $a, b \in \mathbb{N} \setminus \{0\}$ such that $b \geq 2a-1$ and suppose that either $char(\mathbb{K}) = 0$ or $char(\mathbb{K}) \geq a+1$. Let U be a $\binom{b}{a-1}$-dimensional vector space over $\mathbb{K}$ and let B_U be a basis of U whose elements are denoted by the symbols $\bar{e}[i_1 i_2 \ldots i_{a-1}]$ where $i_1, i_2, \ldots, i_{a-1}$ are $a-1$ elements of $\{1,2,\ldots,b\}$ satisfying $i_1 < i_2 < \cdots < i_{a-1}$. Let $\mathcal{A}$ denote the set of all subsets of size a of $\{1,2,\ldots,b\}$ and for every $A = \{i_1, i_2, \ldots, i_a\} \in \mathcal{A}$ with $i_1 < i_2 < \cdots < i_a$, define $\bar{v}_A := \sum_{j=1}^{a} \bar{e}[i_1 \cdots \widehat{i_j} \cdots i_a]$. Then every vector of B_U can be written as a linear combination of the vectors $\bar{v}_A$, $A \in \mathcal{A}$.*

Proof. One readily observes that if the lemma holds for a certain triple $(\mathbb{K}, a, b)$, then it also holds for every triple $(\mathbb{K}, a, b')$ where $b' \geq b$. So, in the sequel we may suppose that $b = 2a-1$.

Let $\bar{e}[i_1^* i_2^* \cdots i_{a-1}^*]$ be an arbitrary vector of B_U. For every $j \in \{0, \ldots, a-1\}$, let $\mathcal{A}_j$ denote the set of all subsets of size a of $\{1,2,\ldots,2a-1\}$ containing $\{i_1^*, \ldots, i_j^*\}$. For every $j \in \{0, \ldots, a-1\}$ distinct elements $n_1, n_2, \ldots, n_j$ of $\{i_1^*, i_2^*, \ldots, i_{a-1}^*\}$ satisfying $n_1 < n_2 < \ldots < n_j$, let $\bar{s}[n_1, n_2, \ldots, n_j]$ denote the sum of all elements $\bar{e}[i_1 i_2 \cdots i_{a-1}]$ where $i_1, i_2, \ldots, i_{a-1}$ are $a-1$ elements of $\{1, \ldots, 2a-1\}$ satisfying $\{n_1, n_2, \ldots, n_j\} \subseteq \{i_1, i_2, \ldots, i_{a-1}\}$ and $i_1 < i_2 < \cdots < i_{a-1}$. If $j = 0$, then the vector $\bar{s}[-]$ will shortly be denoted by $\bar{s}$. If $j = a-1$, then $\bar{s}[n_1, \ldots, n_j] = \bar{s}[i_1^*, \ldots, i_{a-1}^*]$ coincides with the vector $\bar{e}[i_1^*, \ldots, i_{a-1}^*]$. We prove by induction on $j \in \{0, \ldots, a-1\}$ that every vector $\bar{s}[n_1, \ldots, n_j]$ can be written as a linear combination of the vectors $\bar{v}_A$, $A \in \mathcal{A}$.

Suppose first that $j = 0$. Then $\sum_{A \in \mathcal{A}_0} \bar{v}_A = a \cdot \bar{s}$ and hence $\bar{s} = \frac{1}{a} \sum_{A \in \mathcal{A}_0} \bar{v}_A$ since $a \neq 0$.

Suppose $j \in \{1, \ldots, a-1\}$ and that the claim holds for smaller values of j. Without loss of generality, we may suppose that $(n_1, \ldots, n_j) = (i_1^*, \ldots, i_j^*)$. Then $\sum_{A \in \mathcal{A}_j} \bar{v}_A = (a-j) \cdot \bar{s}[i_1^*, \ldots, i_j^*] + \sum_{j'=1}^{j} \bar{s}[i_1^* \ldots \widehat{i_{j'}^*} \cdots i_j^*]$. By the induction hypothesis, $\sum_{j'=1}^{j} \bar{s}[i_1^* \ldots \widehat{i_{j'}^*} \cdots i_j^*]$ can be written as a linear combination of the vectors $\bar{v}_A$, $A \in \mathcal{A}$. Hence, since $a - j \neq 0$, also $\bar{s}[i_1^*, \ldots, i_j^*]$ can be written as a linear combination of the vectors $\bar{v}_A$, $A \in \mathcal{A}$.

For $j = a-1$, we find that the vector $\bar{s}[i_1^*, \ldots, i_{a-1}^*] = \bar{e}[i_1^*, \ldots, i_{a-1}^*]$ can be written as a linear combination of the vectors $\bar{v}_A$, $A \in \mathcal{A}$. $\square$

Lemma 4.5 *Let $k \in \{3, \ldots, n+1\}$, $l \in \{1, \ldots, \min(k-1, n)\}$ and suppose that either $char(\mathbb{K}) = 0$ or $char(\mathbb{K}) \geq \min(k-l+1, \lfloor \frac{k}{2} \rfloor + 1)$. Then $\theta_k(W_{k,l}) = W_{k-2,l-1}$.*

Proof. It suffices to prove that $\bar{v}_1 \wedge \cdots \wedge \bar{v}_{k-2} \in \theta_k(W_{k,l})$ for all linearly independent vectors $\bar{v}_1, \ldots, \bar{v}_{k-2}$ of V such that $\langle \bar{v}_1, \ldots, \bar{v}_{l-1} \rangle$ is totally isotropic with respect to f. By Lemma 2.3, there exists a hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ of V, a $\lambda \in \mathbb{K}^*$ and an $m \in \{\max(0, k-2-n), \ldots, \min(k-l-1, \lfloor \frac{k}{2} \rfloor - 1)\}$ such that $\bar{v}_1 \wedge \cdots \wedge \bar{v}_{k-2} = \lambda \cdot (\bar{e}_1 \wedge \bar{f}_1) \wedge \cdots \wedge (\bar{e}_m \wedge \bar{f}_m) \wedge \bar{e}_{m+1} \wedge \cdots \wedge \bar{e}_{k-2-m}$. Now, $n-(k-2-m) = n+m+2-k \geq m+1$. By Lemma 4.4, $\bar{v}_1 \wedge \cdots \wedge \bar{v}_{k-2}$ belongs to the subspace of $W_{k-2,l-1}$ generated by all vectors $\theta_k\Big((\bar{e}_{i_1} \wedge \bar{f}_{i_1}) \wedge \cdots \wedge (\bar{e}_{i_{m+1}} \wedge \bar{f}_{i_{m+1}}) \wedge \bar{e}_{m+1} \wedge \cdots \wedge \bar{e}_{k-2-m}\Big)$, where $i_1, \ldots, i_{m+1}$ are elements of $\{1, \ldots, m, k-1-m, \ldots, n\}$ satisfying $i_1 < \cdots < i_{m+1}$. The vector $(\bar{e}_{i_1} \wedge \bar{f}_{i_1}) \wedge \cdots \wedge (\bar{e}_{i_{m+1}} \wedge \bar{f}_{i_{m+1}}) \wedge \bar{e}_{m+1} \wedge \cdots \wedge \bar{e}_{k-2-m}$ belongs to $W_{k,k-1-m}$ and hence also to $W_{k,l}$ since $l \leq k-1-m$. □

Lemma 4.6 *Suppose $k \in \{2, \ldots, n\}$ and that either $char(\mathbb{K}) = 0$ or $char(\mathbb{K}) \geq \lfloor \frac{k}{2} \rfloor + 1$. Then $W_{k,k} = ker(\theta_k)$ and $\dim(W_{k,k}) = \binom{2n}{k} - \binom{2n}{k-2}$.*

Proof. Suppose first that $k = 2$. If $B = (\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_n, \bar{f}_n)$ is a hyperbolic basis of V, then by Lemma 2.2, $\bigwedge^2 V = \langle W_{2,2}, \bar{e}_1 \wedge \bar{f}_1 \rangle$. Since $W_{2,2} \subseteq ker(\theta_2)$ and $\bar{e}_1 \wedge \bar{f}_1 \notin ker(\theta_2)$, we have $W_{2,2} = ker(\theta_2)$ and $\dim(W_{2,2}) = \binom{2n}{2} - \binom{2n}{0}$.

Suppose now that $k \geq 3$. By Lemma 4.5, we have $Im(\theta_k) = \theta_k(\bigwedge^k V) = \theta_k(W_{k,1}) = W_{k-2,0} = \bigwedge^{k-2} V$. Hence by Lemma 4.2, $\dim(W_{k,k}) \leq \dim(ker(\theta_k)) = \dim(\bigwedge^k V) - \dim(Im(\theta_k)) = \binom{2n}{k} - \binom{2n}{k-2}$. Lemma 3.2 then implies that $\dim(W_{k,k}) = \binom{2n}{k} - \binom{2n}{k-2}$ and $ker(\theta_k) = W_{k,k}$. □

Lemma 4.7 *If $\mathbb{K}$ is a field of characteristic 0, then $A(n,k,l,\mathbb{K}) = \binom{2n}{k} - \binom{2n}{2l-k-2}$ for every $k \in \{1, \ldots, 2n\}$ and every $l \in \mathbb{N}$ satisfying $0 \leq l \leq \min(n,k)$.*

Proof. We will prove this by induction on k. By Corollary 2.6, the lemma holds if $k \leq 2$ or $l \leq \lceil \frac{k}{2} \rceil$. So, in the sequel we will suppose that $k \geq 3$ and $l > \lceil \frac{k}{2} \rceil$. Since $n \geq l$, this implies that $k \leq 2n-2$.

Suppose first that $k \in \{3, \ldots, n\}$. If $l = k$, then the lemma holds by Lemma 4.6. So, suppose $l \leq k-1$. Consider the restriction $\theta_{k,l} : W_{k,l} \to W_{k-2,l-1}$ of θ_k to the subspace $W_{k,l}$. Then by Lemmas 4.5 and 4.6, $A(n,k,l,\mathbb{K}) = \dim(W_{k,l}) = \dim(Im(\theta_{k,l})) + \dim(ker(\theta_{k,l})) = \dim(W_{k-2,l-1}) + \dim(W_{k,k}) = A(n,k-2,l-1,\mathbb{K}) + \binom{2n}{k} - \binom{2n}{k-2}$. By the induction hypothesis, $A(n,k,l,\mathbb{K}) = \binom{2n}{k-2} - \binom{2n}{2l-k-2} + \binom{2n}{k} - \binom{2n}{k-2} = \binom{2n}{k} - \binom{2n}{2l-k-2}$.

Suppose next that $k \in \{n+1, \ldots, 2n-2\}$. Then by Lemma 4.1 and the induction hypothesis, $A(n,k,l,\mathbb{K}) = A(n, 2n-k, n+l-k, \mathbb{K}) = \binom{2n}{2n-k} - \binom{2n}{2n+2l-2k-2n+k-2} = \binom{2n}{k} - \binom{2n}{2l-k-2}$. □

The following corollary of Lemmas 3.1, 3.2 and 4.7 is precisely Theorem 1.1.

Corollary 4.8 *We have* $A(n,k,l,\mathbb{K}) = \binom{2n}{k} - \binom{2n}{2l-k-2}$ *for every field* $\mathbb{K}$*, for every* $n \in \mathbb{N} \setminus \{0\}$*, for every* $k \in \{1, \dots, 2n\}$ *and every* $l \in \mathbb{N}$ *satisfying* $0 \leq l \leq \min(n,k)$*.*

5 Recursive construction of a basis of $W_{k,l}$

Let $B = (\bar{e}_1, \bar{f}_1, \dots, \bar{e}_n, \bar{f}_n)$ be a hyperbolic basis of V and let $k, l \in \mathbb{N}$ such that $1 \leq k \leq 2n$ and $0 \leq l \leq \min(n,k)$. Since $A(n,k,l,\mathbb{K}) = \binom{2n}{k} - \binom{2n}{2l-k-2}$, the reasoning given in Lemma 3.2 allows us to construct in a recursive way a basis of $W_{k,l}$ which entirely consists of vectors of $\mathcal{G}(B,k,l)$:

- If $l \leq \lceil \frac{k}{2} \rceil$, then a basis of $W_{k,l} = \bigwedge^k V$ is given by the standard basis $\mathcal{B}_k(B)$ of $\bigwedge^k V$ with respect to B.
- If $l = k = 2$, then a basis of $W_{2,2}$ is given in the proof of Lemma 2.2: $\bar{e}_i \wedge \bar{e}_j$, $\bar{f}_i \wedge \bar{f}_j$ $(1 \leq i < j \leq n)$; $\bar{e}_i \wedge \bar{f}_j$ $(i, j \in \{1, \dots, n\}$ with $i \neq j)$; $\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_i \wedge \bar{f}_i$ $(i \in \{2, \dots, n\})$.
- If $k \geq 3$ and $l > \lceil \frac{k}{2} \rceil$, then a basis of $W_{k,l}$ can be computed from a basis of $W'_{k,l}$ (if $l \leq n-1$), a basis of $W'_{k-1,l-1}$ and a suitable basis of $W'_{k-2,l-2}$. (Here, $W'_{k,l}$, $W'_{k-1,l-1}$ and $W'_{k-2,l-2}$ are as defined in the proof of Lemma 3.2).

Example. We will use this inductive construction now to determine a basis of $W_{3,3}$ if $n = 3$.

- Since $n = 3$, there is no contribution of $W'_{3,3}$ to a basis of $W_{3,3}$. Actually, the subspace $W'_{3,3}$ is not defined.
- A basis of $W'_{2,2}$ is given by $\bar{e}_2 \wedge \bar{e}_3$, $\bar{f}_2 \wedge \bar{f}_3$, $\bar{e}_2 \wedge \bar{f}_3$, $\bar{e}_3 \wedge \bar{f}_2$, $\bar{e}_2 \wedge \bar{f}_2 - \bar{e}_3 \wedge \bar{f}_3$. These contribute the following vectors to a basis of $W_{3,3}$: $\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3$, $\bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3$, $\bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3$, $\bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3$, $\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3$, $\bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3$, $\bar{e}_1 \wedge \bar{e}_3 \wedge \bar{f}_2$, $\bar{f}_1 \wedge \bar{e}_3 \wedge \bar{f}_2$, $\bar{e}_1 \wedge (\bar{e}_2 \wedge \bar{f}_2 - \bar{e}_3 \wedge \bar{f}_3)$, $\bar{f}_1 \wedge (\bar{e}_2 \wedge \bar{f}_2 - \bar{e}_3 \wedge \bar{f}_3)$.
- A basis of $W'_{1,1}$ is given by $\bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3$. These contribute the following vectors to $W_{3,3}$: $(\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_3 \wedge \bar{f}_3) \wedge \bar{e}_2$, $(\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_3 \wedge \bar{f}_3) \wedge \bar{f}_2$, $(\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_2 \wedge \bar{f}_2) \wedge \bar{e}_3$, $(\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_2 \wedge \bar{f}_2) \wedge \bar{f}_3$.

So, we have constructed a basis of $W_{3,3}$ of dimension $14 = \binom{6}{3} - \binom{6}{1}$.

Definitions. For every $\alpha = \bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_{j_1}} \wedge \bar{f}_{i'_1} \wedge \cdots \wedge \bar{f}_{i'_{j_2}} \in \mathcal{B}_k(B)$ (so, $j_1 + j_2 = k$), we call (j_1, j_2) the *character* of α. Let $\mathcal{B}_{k,l}(B)$ denote the basis of $W_{k,l}$ which is obtained by applying the above recursive construction, using B as hyperbolic basis. If $\beta \in \mathcal{B}_{k,l}(B)$, then every element of $\mathcal{B}_k(B)$ occurring (with a nonzero factor) in the expansion of β as a linear combination of elements of $\mathcal{B}_k(B)$ has the same character. We call this the *character* of β. Let $B(n,k,l,j_1,j_2)$ denote the number of elements of $\mathcal{B}_{k,l}(B)$ with character (j_1,j_2).

Proposition 5.1 *We have*

$$B(n,k,l,j_1,j_2) \quad = \quad \binom{n}{j_1}\cdot\binom{n}{j_2} - \binom{n}{j_1+l-k-1}\cdot\binom{n}{j_2+l-k-1}$$

for all $n \geq 1$*, all* $k \in \{1,\ldots,2n\}$*, all* $l \in \{0,\ldots,\min(n,k)\}$ *and all* $j_1, j_2 \in \{0,\ldots,k\}$ *with* $j_1 + j_2 = k$*.*

Proof. We will prove this proposition by induction on n. Suppose first that $l \leq \lceil \frac{k}{2} \rceil$. Then $\mathcal{B}_{k,l}(B)$ is the standard basis $\mathcal{B}_k(B)$ of $\bigwedge^k V$ with respect to B. The number of vectors with character (j_1,j_2) in $\mathcal{B}_k(B)$ is equal to $\binom{n}{j_1}\cdot\binom{n}{j_2}$. This is equal to $\binom{n}{j_1}\cdot\binom{n}{j_2} - \binom{n}{j_1+l-k-1}\cdot\binom{n}{j_2+l-k-1}$ since $(j_1+l-k-1)+(j_2+l-k-1) = 2l-k-2 < 0$.

Suppose $k = l = 2$. Then $\mathcal{B}_{2,2}(B) = \{\bar{e}_i \wedge \bar{e}_j, \bar{f}_i \wedge \bar{f}_j \,|\, 1 \leq i < j \leq n\} \cup \{\bar{e}_i \wedge \bar{f}_j \,|\, 1 \leq i,j \leq n,\ i \neq j\} \cup \{\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_i \wedge \bar{f}_i \,|\, 2 \leq i \leq n\}$. Hence, $B(n,2,2,0,2) = \frac{n(n-1)}{2} = \binom{n}{0}\cdot\binom{n}{2} - \binom{n}{-1}\cdot\binom{n}{1}$, $B(n,2,2,1,1) = n(n-1)+n-1 = n^2-1 = \binom{n}{1}\cdot\binom{n}{1} - \binom{n}{0}\cdot\binom{n}{0}$ and $B(n,2,2,2,0) = \frac{n(n-1)}{2} = \binom{n}{2}\cdot\binom{n}{0} - \binom{n}{1}\cdot\binom{n}{-1}$.

Finally, suppose that $k \geq 3$ and $l > \lceil \frac{k}{2} \rceil$. Then $n \geq 3$. By the recursive construction, $B(n,k,l,j_1,j_2) = B(n-1,k-2,l-2,j_1-1,j_2-1) + B(n-1,k-1,l-1,j_1-1,j_2) + B(n-1,k-1,l-1,j_1,j_2-1) + B'(n-1,k,l,j_1,j_2)$, where $B'(n-1,k,l,j_1,j_2)$ is equal to $B(n-1,k,l,j_1,j_2)$ if $n > l$ and equal to 0 otherwise. By the induction hypothesis, we know the precise values of $B(n-1,k-2,l-2,j_1-1,j_2-1)$, $B(n-1,k-1,l-1,j_1-1,j_2)$, $B(n-1,k-1,l-1,j_1,j_2-1)$ and $B'(n-1,k,l,j_1,j_2)$. Notice that the precise value of $B'(n-1,k,l,j_1,j_2)$ is always equal to $\binom{n-1}{j_1}\cdot\binom{n-1}{j_2} - \binom{n-1}{j_1+l-k-1}\cdot\binom{n-1}{j_2+l-k-1}$, also in the case where $l = n$ (recall that $j_1 + j_2 = k$). So, we can compute $B(n,k,l,j_1,j_2)$. Using Pascal's identity a few times, we readily find $B(n,k,l,j_1,j_2) = \binom{n}{j_1}\cdot\binom{n}{j_2} - \binom{n}{j_1+l-k-1}\cdot\binom{n}{j_2+l-k-1}$. $\square$

By Proposition 5.1, we can write the vector space $W_{k,l}$ as a direct sum

$$W_{k,l}(0,k) \oplus W_{k,l}(1,k-1) \oplus \cdots \oplus W_{k,l}(k,0),$$

where $W_{k,l}(j_1,j_2)$ denotes the subspace of $W_{k,l}$ generated by all vectors of $\mathcal{B}_{k,l}(B)$ with character (j_1,j_2). The dimension of the subspace $W_{k,l}$ is given in Proposition 5.1.

In the special case that $k = l = n$, the above decomposition of the space $W_{n,n}$ coincides with the decomposition of the natural embedding space of the symplectic dual polar space $DSp(2n, \mathbb{K})$ as stated in the main theorem of [4].

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